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The growth rates for pure Artin groups of dihedral type

Michihiko Fujii*

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Abstract

We consider the kernel of the natural projection from the Artin group of dihedral type $I_2(k)$ to the associated Coxeter group, which we call a pure Artin group of dihedral type and write $P_{I_2(k)}$. We show that the growth rates for both the spherical growth series and geodesic growth series of $P_{I_2(k)}$ with respect to a natural generating set are Pisot numbers.

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1 Introduction

Let G be a group with a finite generating set. Then there exists a finite generating set Γ of G that is symmetric (i.e., $\Gamma = \Gamma^{-1}$). Let Γ^* be the free monoid generated by Γ . The elements of Γ^* are called *words over Γ* . For each element $g \in G$, we define the length of g with respect to Γ to be the smallest integer $m \geq 0$ for which there exists a word $s_1 \cdots s_m \in \Gamma^*$ such that $g = s_1 \cdots s_m$ in G . Then, the *spherical growth series* is defined as

$$\mathcal{S}_{(G,\Gamma)}(t) := \sum_{n=0}^{\infty} \alpha_n t^n,$$

where α_n for $n \in \mathbf{N} \cup \{0\}$ is the number of elements in G whose lengths are equal to n . Analogously to $\mathcal{S}_{(G,\Gamma)}(t)$, the *geodesic growth series* is defined as

$$\mathcal{G}_{(G,\Gamma)}(t) := \sum_{n=0}^{\infty} \tilde{\alpha}_n t^n,$$

where $\tilde{\alpha}_n$ for $n \in \mathbf{N} \cup \{0\}$ is the number of geodesic words of length n . Recall that a word over Γ is *geodesic* if the corresponding path in the Cayley graph of G with respect to Γ is a

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minimal length edge path joining its endpoints. The *growth rates* for $\mathcal{S}_{(G,\Gamma)}(t)$ and $\mathcal{G}_{(G,\Gamma)}(t)$ are defined as

$$\begin{aligned}\tau_{\mathcal{S}} &:= \limsup_{n \rightarrow \infty} \sqrt[n]{\alpha_n}, \\ \tau_{\mathcal{G}} &:= \limsup_{n \rightarrow \infty} \sqrt[n]{\tilde{\alpha}_n},\end{aligned}$$

respectively. We also call $\tau_{\mathcal{S}}$ and $\tau_{\mathcal{G}}$ the *spherical growth rate* and the *geodesic growth rate* for the pair (G, Γ) , respectively. By the Cauchy-Hadamard theorem, the radius of convergence $R_{\mathcal{S}}$ (resp., $R_{\mathcal{G}}$) of the series $\mathcal{S}_{(G,\Gamma)}(t)$ (resp., $\mathcal{G}_{(G,\Gamma)}(t)$) is the reciprocal of $\tau_{\mathcal{S}}$ (resp., $\tau_{\mathcal{G}}$). Because the group G is a quotient of the free group generated by Γ , it is immediately seen that $R_{\mathcal{S}}$ and $R_{\mathcal{G}}$ are greater than or equal to $1/(\#\Gamma - 1)$ (> 0), and this implies that both $\mathcal{S}_{(G,\Gamma)}(t)$ and $\mathcal{G}_{(G,\Gamma)}(t)$ are holomorphic functions near the origin, $0 \in \mathbb{C}$ (for example, see [3] or [4]).

In this paper, for each integer $k \geq 3$, we consider the *pure Artin group* $P_{I_2(k)}$, which is the kernel of the projection from the Artin group of dihedral type to the associated Coxeter group. This group is geometrically realized as the fundamental group of the complement of some arrangement of k complex lines in \mathbb{C}^2 , which has a natural generating set A consisting of $2k$ elements (see [13] or the presentation (1) in §2 of this paper). In particular, in the case $k = 3$, $P_{I_2(3)}$ is the pure braid group with three strands, and A is its standard generating set. Now, we choose the generating set A and consider the spherical growth series $\mathcal{S}_{(P_{I_2(k)}, A)}(t)$ and the geodesic growth series $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$ for the pair $(P_{I_2(k)}, A)$. Let $\tau_{\mathcal{S}}(k)$ and $\tau_{\mathcal{G}}(k)$ be the spherical growth rate and the geodesic growth rate for the pair $(P_{I_2(k)}, A)$, respectively, and let $R_{\mathcal{S}}(k)$ and $R_{\mathcal{G}}(k)$ be the radii of convergence of $\mathcal{S}_{(P_{I_2(k)}, A)}(t)$ and $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$, respectively. In the previous papers [6] and [8], we presented rational function expressions for $\mathcal{S}_{(P_{I_2(k)}, A)}(t)$ and $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$. In the present paper, using these concrete expressions, we determine $R_{\mathcal{S}}(k)$ and $R_{\mathcal{G}}(k)$ as certain algebraic numbers, and prove that $t = R_{\mathcal{S}}(k)$ and $t = R_{\mathcal{G}}(k)$ are real poles of $\mathcal{S}_{(P_{I_2(k)}, A)}(t)$ and $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$, respectively (see Theorem 3.2). Moreover, we demonstrate the property that $\tau_{\mathcal{S}}(k)$ and $\tau_{\mathcal{G}}(k)$ are Pisot-Vijayaraghavan numbers (see Corollary 3.4). A real algebraic integer $\tau > 1$ is called a *Pisot-Vijayaraghavan number* if all of its algebraic conjugates other than τ itself lie in the unit disk. In the remainder of the paper, for brevity, we use the term *Pisot number* instead of *Pisot-Vijayaraghavan number*.

Before ending this introduction, we comment on previous results concerning number-theoretic properties of the spherical growth rates of finitely generated groups. Floyd [5] proved that the spherical growth rates of two-dimensional non-compact hyperbolic Coxeter groups with respect to the standard generating sets are Pisot numbers. Later, Komori and Umemoto [11] showed that the spherical growth rates of certain three-dimensional non-compact hyperbolic Coxeter groups with respect to the standard generating sets are Pisot numbers. (In the proof of Lemma 3.1(i), we use an argument similar to that given in Lemma 2 of [11].) In fact, the work [11] was motivated by the following conjecture of Kellerhals and Perren [9]: The spherical growth rate of a hyperbolic Coxeter group with respect to the standard generating set is a Perron number, i.e., a real algebraic integer $\tau > 1$ whose algebraic conjugates other than τ itself all have smaller absolute values than τ . Note that a Pisot number is a Perron number by definition. This conjecture is based on the results

of [1], [2] and [12], and Kellerhals and Perren themselves confirmed the validity of their conjecture in the case of certain four-dimensional cocompact hyperbolic Coxeter groups in [9]. Recently, motivated by Floyd's work [5], Kolpakov [10] showed that the spherical growth rate of a certain three-dimensional non-compact hyperbolic Coxeter group, which is obtained through a deformation of three-dimensional cocompact hyperbolic Coxeter groups, is a Pisot number.

2 Growth series for pure Artin groups of dihedral type

In this section, we present definitions and basic facts concerning growth series for the pure Artin groups of dihedral type (see [6], [7] and [8] for details).

Let k be an integer greater than 2, and let $G_{I_2(k)}$ be the *Artin group of dihedral type $I_2(k)$* and $\overline{G}_{I_2(k)}$ be the *Coxeter group of dihedral type $I_2(k)$* , which are defined by

$$\begin{aligned} G_{I_2(k)} &:= \langle \sigma_1, \sigma_2 \mid \langle \sigma_1 \sigma_2 \rangle^k = \langle \sigma_2 \sigma_1 \rangle^k \rangle, \\ \overline{G}_{I_2(k)} &:= \langle \sigma_1, \sigma_2 \mid \langle \sigma_1 \sigma_2 \rangle^k = \langle \sigma_2 \sigma_1 \rangle^k, \sigma_1^2 = \sigma_2^2 = 1 \rangle, \end{aligned}$$

respectively, where

$$\langle \sigma_i \sigma_j \rangle^k := \underbrace{\sigma_i \sigma_j \sigma_i \sigma_j \sigma_i \cdots}_{k \text{ letters}}.$$

The group $\overline{G}_{I_2(k)}$ is the dihedral group of order $2k$. Let $\sigma := \sigma_1 \sigma_2$ and $\tau := \sigma_2$. Then we have the usual presentation of the dihedral group:

$$\langle \sigma, \tau \mid \sigma^k = \tau^2 = (\sigma\tau)^2 = 1 \rangle.$$

Next, note that there is a natural surjective homomorphism

$$p : G_{I_2(k)} \rightarrow \overline{G}_{I_2(k)}.$$

We call the kernel of p the *pure Artin group of dihedral type* and denote it by $P_{I_2(k)}$. The group $P_{I_2(k)}$ has the following presentation:

$$P_{I_2(k)} = \langle a_1, \dots, a_k \mid a_1 \cdots a_k = a_2 \cdots a_k a_1 = a_3 \cdots a_k a_1 a_2 = \cdots = a_k a_1 \cdots a_{k-1} \rangle. \quad (1)$$

In this paper, we consider the generating set

$$A = \{a_1, \dots, a_k, a_1^{-1}, \dots, a_k^{-1}\}$$

for the group $P_{I_2(k)}$.

Let A^* denote the free monoid generated by A . We refer to the elements of A^* as *words over A* . Let ε be the *null* word, which is the identity of the monoid A^* . The length of a word w is the number of letters it contains, which is denoted by $|w|$. The length of ε is zero.

Since A generates the group $P_{I_2(k)}$, there exists a natural surjective monoid homomorphism $\pi : A^* \rightarrow P_{I_2(k)}$. A word $w \in \pi^{-1}(g)$ is called a *representative* of g . The length of a group element g is regarded as the quantity

$$\|g\| = \min\{|w| \mid w \in \pi^{-1}(g)\}.$$

A word $w \in A^*$ for which the relation $|w| = \|\pi(w)\|$ holds is termed a *geodesic*. A word $w_1 \cdots w_m \in A^*$ is called a *reduced* word if $w_i \neq w_{i+1}^{-1}$ for all $i \in \{1, \dots, m-1\}$. A geodesic representative is a reduced word.

The *spherical growth series* and *geodesic growth series* for the pair $(P_{I_2(k)}, A)$ are defined by the formal power series

$$\begin{aligned} \mathcal{S}_{(P_{I_2(k)}, A)}(t) &:= \sum_{n=0}^{\infty} \alpha_n t^n, \\ \mathcal{G}_{(P_{I_2(k)}, A)}(t) &:= \sum_{n=0}^{\infty} \tilde{\alpha}_n t^n, \end{aligned}$$

respectively, where for each $n \in \mathbf{N} \cup \{0\}$, we define

$$\begin{aligned} \alpha_n &:= \#\{g \in P_{I_2(k)} \mid \|g\| = n\}, \\ \tilde{\alpha}_n &:= \#\{w \in A^* \mid |w| = \|\pi(w)\| = n\}. \end{aligned}$$

Next, we consider the growth rates of the above series:

$$\begin{aligned} \tau_{\mathcal{S}}(k) &:= \limsup_{n \rightarrow \infty} \sqrt[n]{\alpha_n}, \\ \tau_{\mathcal{G}}(k) &:= \limsup_{n \rightarrow \infty} \sqrt[n]{\tilde{\alpha}_n}. \end{aligned}$$

We call $\tau_{\mathcal{S}}(k)$ and $\tau_{\mathcal{G}}(k)$ the *spherical growth rate* and the *geodesic growth rate* for the pair $(P_{I_2(k)}, A)$, respectively. Note that the relation $\tau_{\mathcal{S}}(k) \leq \tau_{\mathcal{G}}(k)$ follows from the above definitions. Let $R_{\mathcal{S}}(k)$ and $R_{\mathcal{G}}(k)$ denote the radii of convergence of the series $\mathcal{S}_{(P_{I_2(k)}, A)}(t)$ and $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$, respectively. By the Cauchy-Hadamard theorem, we have

$$\begin{aligned} R_{\mathcal{S}}(k) &= \frac{1}{\tau_{\mathcal{S}}(k)}, \\ R_{\mathcal{G}}(k) &= \frac{1}{\tau_{\mathcal{G}}(k)}. \end{aligned}$$

As described in the introduction, both $\mathcal{R}_{\mathcal{S}}(k)$ and $\mathcal{R}_{\mathcal{G}}(k)$ are greater than or equal to $1/(2k-1)$, and this implies that both $\mathcal{S}_{(P_{I_2(k)}, A)}(t)$ and $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$ are holomorphic functions near the origin, $0 \in \mathbf{C}$.

Now, we present the rational function expressions for $\mathcal{S}_{(P_{I_2(k)}, A)}(t)$ and $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$, which are given in [6] and [8]. In order to simplify the presentation of the growth series, we introduce the following polynomials in the variable t :

$$\begin{cases} T_0 &:= 0, \\ T_n &:= t + t^2 + \cdots + t^n \quad \text{for } n \in \mathbf{N}. \end{cases}$$

Next, we define the following polynomials in the variable t :

$$\begin{cases} f_0(t) := 1 - t, \\ f_i(t) := 1 - (k-1)(T_{i-1} + T_{k-i}) & \text{for } i \in \{1, \dots, k\}, \\ \\ g_0(t) := 1 - kt, \\ g_i(t) := 1 - (k-1)(T_i + T_{k-i}) & \text{for } i \in \{1, \dots, k-1\}. \end{cases}$$

Using these polynomials, the theorems given in [6] and [8] can be expressed as follows:

Theorem 2.1 ([6, 8]) *The spherical and geodesic growth series for the pure Artin group $P_{I_2(k)}$ of dihedral type with respect to the generating set A possess the following rational function expressions:*

$$\begin{aligned} \mathcal{S}_{(P_{I_2(k)}, A)}(t) &= \frac{1+t^k}{f_0(t)f_1(t)} + \frac{k}{2} \sum_{p=1}^{k-1} \frac{t^{k-p} + t^p}{f_p(t)f_{p+1}(t)}, \\ \mathcal{G}_{(P_{I_2(k)}, A)}(t) &= \frac{2kt^k}{g_0(t)f_1(t)} + \sum_{p=1}^{k-1} \frac{1+T_p+T_{k-p}}{g_p(t)} - \sum_{p=1}^{k-2} \frac{1+T_p+T_{k-1-p}}{f_{p+1}(t)}. \end{aligned} \tag{2}$$

3 Poles and growth rates of the growth series

In this section, we determine the radii of convergence $\mathcal{R}_S(k)$ and $\mathcal{R}_G(k)$ introduced in the previous section, and show that $t = R_S(k)$ and $t = R_G(k)$ are real poles of $\mathcal{S}_{(P_{I_2(k)}, A)}(t)$ and $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$, respectively. Moreover, we obtain the property that the growth rates $\tau_S(k)$ and $\tau_G(k)$ are Pisot numbers, i.e., a real algebraic integer $\tau > 1$ whose algebraic conjugates other than τ itself lie in the unit disk.

We first prove the following lemma.

Lemma 3.1 *Let $k \geq 3$ be an integer. Then the following hold.*

(i) *For each $a, b \in \mathbf{N}$, the equation*

$$1 - (k-1)(T_a + T_b) = 0$$

has a unique positive real solution $\rho_{a,b}(k)$ contained in the open interval $(\frac{1}{2k-1}, \frac{1}{k})$. Moreover, the absolute value of any other solution is greater than 1.

(ii) *For each $a \in \mathbf{N}$, the equation*

$$1 - (k-1)(T_a + T_0) = 0$$

has a unique positive real solution $\rho_{a,0}(k)$ contained in the open interval $(\frac{1}{k}, \frac{2}{k})$. Moreover, the absolute value of any other solution is greater than 1.

(iii) Let $m \geq 1$ be an integer. Then, for each $a, b \in \mathbf{N} \cup \{0\}$ satisfying $a + b = m$ and $a \leq b$, we have the following:

(iii-1) If m is odd and $(a, b) \neq (\frac{m-1}{2}, \frac{m-1}{2} + 1)$, then $\rho_{\frac{m-1}{2}, \frac{m-1}{2} + 1}(k) < \rho_{a,b}(k)$.

(iii-2) If m is even and $(a, b) \neq (\frac{m}{2}, \frac{m}{2})$, then $\rho_{\frac{m}{2}, \frac{m}{2}}(k) < \rho_{a,b}(k)$.

(iv) For each $a, b \in \mathbf{N} \cup \{0\}$ with $a + b \geq 1$, we have

$$\begin{cases} \rho_{a+1,b}(k) < \rho_{a,b}(k), \\ \rho_{a,b+1}(k) < \rho_{a,b}(k). \end{cases}$$

Proof. (i) Define $f(t) := 1 - (k-1)(T_a + T_b)$, $h(t) := (1-t)f(t)$, $h_1(t) := (k-1)(t^{a+1} + t^{b+1})$ and $h_2(t) := 1 - (2k-1)t$. Then we have

$$h(t) = (1-t)f(t) = (k-1)(t^{a+1} + t^{b+1}) + 1 - (2k-1)t = h_1(t) + h_2(t). \quad (3)$$

Also, because $h(1) = 0$ and $h'(1) > 0$, the following holds:

$$h(r) < 0 \quad \text{for any } r \in \left(\frac{1}{2k-1}, 1\right) \text{ sufficiently close to 1.} \quad (4)$$

Choose $r \in (\frac{1}{2k-1}, 1)$ sufficiently close to 1. Then, for any complex number t on the circle $\{z \in \mathbf{C} \mid |z| = r\}$, we have

$$\begin{aligned} |h_1(t)| = (k-1)|t^{a+1} + t^{b+1}| &\leq (k-1)(r^{a+1} + r^{b+1}) \\ &< (2k-1)r - 1 \quad (\text{from (4)}) \\ &\leq |(2k-1)t - 1| = |h_2(t)|. \end{aligned} \quad (5)$$

Hence, from (3), (5) and Rouché's theorem, $h(t)$ also has a unique zero in the disk $\{z \in \mathbf{C} \mid |z| < r\}$, because $h_2(t) = 0$ has the unique solution $t = \frac{1}{2k-1}$ in the disk $\{z \in \mathbf{C} \mid |z| < r\}$. This holds for any $r < 1$ sufficiently close to 1. Thus, $h(t)$, and hence $f(t)$, has a unique zero ρ in the open unit disk $\{z \in \mathbf{C} \mid |z| < 1\}$.

Next, suppose that $f(t)$ has a zero on the unit circle $\{z \in \mathbf{C} \mid |z| = 1\}$. Then there exists $\theta \in \mathbf{R}$ such that $f(e^{i\theta}) = 0$. From (3), we also have $h(e^{i\theta}) = 0$. This implies that

$$(k-1)|e^{i(a+1)\theta} + e^{i(b+1)\theta}| = |(2k-1)e^{i\theta} - 1|. \quad (6)$$

Furthermore, from the triangle inequality, we have

$$\begin{aligned} 2(k-1) &\geq (k-1)|e^{i(a+1)\theta} + e^{i(b+1)\theta}|, \\ |(2k-1)e^{i\theta} - 1| &\geq 2k-2. \end{aligned} \quad (7)$$

From (6) and (7), we obtain $|(2k-1)e^{i\theta} - 1| = 2k-2$, and this implies that $e^{i\theta} = 1$. However, this contradicts the relation $f(1) \neq 0$, which follows from the definition. Hence, $f(t)$ has no zeros on the unit circle $\{z \in \mathbf{C} \mid |z| = 1\}$. Thus, any zero of $f(t)$ other than ρ must be contained in $\{z \in \mathbf{C} \mid |z| > 1\}$.

Finally, we show that the unique zero ρ within $\{z \in \mathbf{C} \mid |z| < 1\}$ lies in the open interval $(\frac{1}{2k-1}, \frac{1}{k})$. From the assumption $k \geq 3$ and the fact that $a, b \geq 1$, it is readily confirmed that

$f(\frac{1}{k}) < 0$. Next, from (3), we obtain $h(\frac{1}{2k-1}) > 0$, and this implies that $f(\frac{1}{2k-1}) > 0$. Hence, because $f(t)$ is a strictly monotonically decreasing function of t on the open interval $(0, 1)$, there exists a unique zero $\rho_{a,b}(k)$ in $(\frac{1}{2k-1}, \frac{1}{k})$; this zero coincides with ρ .

(ii) The assertion follows from an argument similar to that given in (i).

(iii) Consider the case that m is even. First, note that the two non-negative integers a and b satisfy $a < \frac{m}{2} < b$, since $a + b = m$, $a \leq b$ and $(a, b) \neq (\frac{m}{2}, \frac{m}{2})$. Now, define the real, positive numbers ρ_1 and ρ_0 as $\rho_1 := \rho_{\frac{m}{2}, \frac{m}{2}}(k)$ and $\rho_0 := \rho_{a,b}(k)$. Then these satisfy the equalities

$$\begin{aligned} 1 - (k-1)(\rho_1 + \cdots + \rho_1^{\frac{m}{2}} + \rho_1 + \cdots + \rho_1^{\frac{m}{2}}) &= 0, \\ 1 - (k-1)(\rho_0 + \cdots + \rho_0^a + \rho_0 + \cdots + \rho_0^b) &= 0. \end{aligned}$$

From these equalities and the conditions $a + b = m$ and $a < \frac{m}{2} < b$, we obtain

$$\begin{aligned} &(\rho_1 - \rho_0) + \cdots + (\rho_1^{\frac{m}{2}} - \rho_0^{\frac{m}{2}}) + (\rho_1 - \rho_0) + \cdots + (\rho_1^a - \rho_0^a) \\ &= (\rho_0^{\frac{m}{2}+1} + \cdots + \rho_0^b) - (\rho_1^{a+1} + \cdots + \rho_1^{\frac{m}{2}}) \\ &= (\rho_0^{\frac{m}{2}+1} - \rho_1^{a+1}) + (\rho_0^{\frac{m}{2}+2} - \rho_1^{a+2}) + \cdots + (\rho_0^b - \rho_1^{\frac{m}{2}}). \end{aligned} \quad (8)$$

Next, suppose that $\rho_0 \leq \rho_1$. Then the leftmost side of (8) is positive or zero. On the other hand, because $\frac{m}{2} + 1 > a + 1$, $\frac{m}{2} + 2 > a + 2, \dots, b > \frac{m}{2}$ and $0 < \rho_0 \leq \rho_1 < 1$, the rightmost side of (8) is negative. This is a contradiction. Thus, $\rho_0 > \rho_1$. The case in which m is odd is proved by a similar argument.

(iv) With $\rho_1 := \rho_{a+1,b}(k)$ and $\rho_0 := \rho_{a,b}(k)$, we have the equalities

$$\begin{aligned} 1 - (k-1)(\rho_1 + \cdots + \rho_1^a + \rho_1^{a+1} + \rho_1 + \cdots + \rho_1^b) &= 0, \\ 1 - (k-1)(\rho_0 + \cdots + \rho_0^a + \rho_0 + \cdots + \rho_0^b) &= 0. \end{aligned}$$

From these equalities, we obtain

$$\sum_{d=1}^a (\rho_1^d - \rho_0^d) + \sum_{d=1}^b (\rho_1^d - \rho_0^d) = -\rho_1^{a+1}. \quad (9)$$

Because both ρ_1 and ρ_0 are positive real numbers, from (9), we have $\rho_1 < \rho_0$, that is, $\rho_{a+1,b}(k) < \rho_{a,b}(k)$. The second inequality is shown similarly. \square

Now, we consider the following polynomials:

$$\begin{aligned} E(t) &:= \begin{cases} f_0(t)f_1(t) \cdots f_{\frac{k+1}{2}}(t) & \text{if } k \text{ is odd,} \\ f_0(t)f_1(t) \cdots f_{\frac{k}{2}-1}(t)\{f_{\frac{k}{2}}(t)\}^2 & \text{if } k \text{ is even,} \end{cases} \\ G(t) &:= \begin{cases} g_0(t)g_1(t) \cdots g_{\frac{k-1}{2}}(t)f_1(t) \cdots f_{\frac{k+1}{2}}(t) & \text{if } k \text{ is odd,} \\ g_0(t)g_1(t) \cdots g_{\frac{k}{2}}(t)f_1(t) \cdots f_{\frac{k}{2}}(t) & \text{if } k \text{ is even.} \end{cases} \end{aligned} \quad (10)$$

The polynomials $E(t)$ and $G(t)$ are, respectively, common denominators of the terms of $\mathcal{S}_{(P_{I_2(k)}, A)}(t)$ and $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$ given in (2). Next, we rewrite the expressions appearing in (2)

using these common denominators and sum the terms. We thereby obtain single fraction expressions for $\mathcal{S}_{(P_{I_2(k)}, A)}(t)$ and $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$. Writing the corresponding numerators as $F(t)$ and $H(t)$, we have

$$\mathcal{S}_{(P_{I_2(k)}, A)}(t) = \frac{F(t)}{E(t)}, \quad \mathcal{G}_{(P_{I_2(k)}, A)}(t) = \frac{H(t)}{G(t)}. \quad (11)$$

Considering these single fraction expressions, we obtain the main result of this paper:

Theorem 3.2 *Let $k \geq 3$ be an integer. Also, let $\mathcal{R}_S(k)$ and $\mathcal{R}_G(k)$ be the radii of convergence of the spherical growth series $\mathcal{S}_{(P_{I_2(k)}, A)}(t)$ and the geodesic growth series $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$, respectively. Then the following hold:*

(i)

$$\begin{aligned} \mathcal{R}_S(k) &= \begin{cases} \rho_{\frac{k-1}{2}, \frac{k-1}{2}}(k) & \text{if } k \text{ is odd,} \\ \rho_{\frac{k}{2}-1, \frac{k}{2}}(k) & \text{if } k \text{ is even,} \end{cases} \\ \mathcal{R}_G(k) &= \begin{cases} \rho_{\frac{k-1}{2}, \frac{k+1}{2}}(k) & \text{if } k \text{ is odd,} \\ \rho_{\frac{k}{2}, \frac{k}{2}}(k) & \text{if } k \text{ is even,} \end{cases} \end{aligned}$$

where $\rho_{a,b}(k)$ ($a, b \in \mathbf{N}$) is the positive real number given in Lemma 3.1(i).

(ii) *The points $t = \mathcal{R}_S(k)$ and $t = \mathcal{R}_G(k)$ are real poles of $\mathcal{S}_{(P_{I_2(k)}, A)}(t)$ and $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$, respectively.*

(iii)

$$\frac{1}{2k-1} < \mathcal{R}_G(k) < \mathcal{R}_S(k) < \frac{1}{k}.$$

(iv) *The sequences $\{\mathcal{R}_S(k)\}_{k \geq 3}$ and $\{\mathcal{R}_G(k)\}_{k \geq 3}$ are strictly decreasing.*

Proof. We only consider the case in which k is odd, because the proof for the case in which k is even can be carried out similarly.

(i) and (ii) By Lemmas 3.1(i) and (ii), for each $i \in \{1, \dots, \frac{k+1}{2}\}$, the polynomial $f_i(t)$ has a unique zero $\rho_{i-1, k-i}(k)$ in the open interval $(0, 1)$. Moreover, by Lemma 3.1(iii-2), $\rho_{\frac{k-1}{2}, \frac{k-1}{2}}(k)$, which is such a zero for $f_{\frac{k+1}{2}}(t)$, satisfies $\rho_{\frac{k-1}{2}, \frac{k-1}{2}}(k) < \rho_{i-1, k-i}(k)$ for all $i \in \{1, \dots, \frac{k-1}{2}\}$. Thus, the numerator $F(t)$ of $\mathcal{S}_{(P_{I_2(k)}, A)}(t)$ does not have a zero at $t = \rho_{\frac{k-1}{2}, \frac{k-1}{2}}(k)$. Hence, $t = \rho_{\frac{k-1}{2}, \frac{k-1}{2}}(k)$ is a pole of $\mathcal{S}_{(P_{I_2(k)}, A)}(t)$.

Furthermore, again from Lemmas 3.1(i) and (ii), the other zeros of $f_i(t)$ ($1 \leq i \leq \frac{k+1}{2}$) are contained in the exterior of the unit disk. Hence, among all the poles of $\mathcal{S}_{(P_{I_2(k)}, A)}(t)$, $t = \rho_{\frac{k-1}{2}, \frac{k-1}{2}}(k)$ is that nearest to the origin, $0 \in \mathbf{C}$. This implies that $\mathcal{R}_S(k) = \rho_{\frac{k-1}{2}, \frac{k-1}{2}}(k)$.

Similarly, with the aid of Lemmas 3.1(iii-1) and (iv), we can prove that the zero $\rho_{\frac{k-1}{2}, \frac{k+1}{2}}(k)$ for the polynomial $g_{\frac{k-1}{2}}(t)$ is a pole of $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$ and that among all the poles of $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$, $t = \rho_{\frac{k-1}{2}, \frac{k+1}{2}}(k)$ is that nearest to the origin. This implies that $\mathcal{R}_G(k) = \rho_{\frac{k-1}{2}, \frac{k+1}{2}}(k)$.

(iii) The assertion is immediately deduced from part (i) and Lemma 3.1(i).

(iv) Because we are considering the case in which k is odd, from part (i), we have $\mathcal{R}_S(k) = \rho_{\frac{k-1}{2}, \frac{k-1}{2}}(k)$ and $\mathcal{R}_S(k+1) = \rho_{\frac{k-1}{2}, \frac{k+1}{2}}(k+1)$. Define $\rho_0 := \rho_{\frac{k-1}{2}, \frac{k-1}{2}}(k)$ and $\rho_1 := \rho_{\frac{k-1}{2}, \frac{k+1}{2}}(k+1)$. Then we have

$$\begin{aligned} 1 - (k-1)(\rho_0 + \cdots + \rho_0^{\frac{k-1}{2}} + \rho_0 + \cdots + \rho_0^{\frac{k-1}{2}}) &= 0, \\ 1 - k(\rho_1 + \cdots + \rho_1^{\frac{k-1}{2}} + \rho_1 + \cdots + \rho_1^{\frac{k-1}{2}} + \rho_1^{\frac{k+1}{2}}) &= 0. \end{aligned}$$

From these equalities, we obtain

$$2 \left((k\rho_1 - (k-1)\rho_0) + \cdots + (k\rho_1^{\frac{k-1}{2}} - (k-1)\rho_0^{\frac{k-1}{2}}) \right) = -k\rho_1^{\frac{k+1}{2}}. \quad (12)$$

Because both ρ_1 and ρ_0 are positive real numbers, from (12), we have $\rho_1 < \rho_0$, that is, $\mathcal{R}_S(k+1) < \mathcal{R}_S(k)$. The inequality $\mathcal{R}_G(k+1) < \mathcal{R}_G(k)$ is shown similarly. \square

Example 3.3

$$\begin{cases} R_S(3) = \frac{1}{4}, R_S(4) = \frac{-3+2\sqrt{3}}{3}, R_S(5) = \frac{-2+\sqrt{6}}{4}, \\ R_G(3) = \frac{-2+\sqrt{6}}{2}, R_G(4) = \frac{-3+\sqrt{15}}{6}. \end{cases}$$

From Theorem 3.2 and Lemma 3.1(i), we obtain

Corollary 3.4 *Let $k \geq 3$ be an integer. Then the spherical growth rate $\tau_S(k)$ and the geodesic growth rate $\tau_G(k)$ for the pair $(P_{I_2(k)}, A)$ satisfy the following:*

- (i) *Both $\tau_S(k)$ and $\tau_G(k)$ are Pisot numbers.*
- (ii) *$k < \tau_S(k) < \tau_G(k) < 2k - 1$.*
- (iii) *The sequences $\{\tau_S(k)\}_{k \geq 3}$ and $\{\tau_G(k)\}_{k \geq 3}$ are strictly increasing.*

Remark. From Theorem 3.2 of [8], the polynomials $G(t)$ and $H(t)$ appearing in (11) do not have a common zero. By contrast, it is not known whether $E(t)$ and $F(t)$ in (11) have a common zero or not. We conjecture that they do not. With regard to this conjecture, from the arguments given in the proof of Theorem 3.2, we know that while the denominator, $E(t)$, does have a zero at $t = \mathcal{R}_S(k)$, the numerator, $F(t)$, does not.

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